Indian Statistical Institute, Bangalore
M.Math II year 2018-2019
Semester II : Operator Theory

Final Exam — Date: 25.04.2019

Maximum Marks: 100 — Duration: 3 hours

Note: Any score above 100 will be taken as 100. State the results very clearly that you are using in your answers.

1. (15) Let V be the vector space of complex-valued functions on $\mathbb{Z}$. Give an example of a linear operator $T : V \to V$ such that, for any given $\lambda \in \mathbb{C}$, the equation $Tx = \lambda x$ has a non-zero solution in V.

2. (15) Let K be a compact subset of $\mathbb{C}$. Give an example of a bounded linear operator $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ such that the spectrum of T is K.

3. (20) Let $\{e_n\}_{n\in\mathbb{Z}}$ be the set of all orthonormal basis vectors of $l^2(\mathbb{Z})$ and S be the shift operator on $l^2(\mathbb{Z})$ satisfying $Se_n = e_{n+1}$ for all $n \in \mathbb{Z}$. Find a measure μ on real line such that for all $k = 0, 1, 2, \ldots$
$$\langle (e_0 + e_1), (S + S^*)^k (e_0 + e_1) \rangle = \int x^k d\mu(x).$$

4. (5+10+15) Consider the C^*-algebra $M(n)$, the set of all $n \times n$ complex matrices, and a linear functional ϕ on $M(n)$
   (a) Show that there exists $B \in M(n)$ such that $\phi(A) = tr(AB)$, for all $A \in M(n)$.
   (b) Show that ϕ is a state if and only if B is a positive definite matrix.
   (c) Show that the identity element, as well as any unitary element, of $M(n)$ is an extreme point of the closed unit ball of $M(n)$.

5. (15) Consider the operator T on $\mathbb{L}^2[0, 1]$ given by
$$(Tf)(x) = \int_0^x f(y)dy.$$
Show that T is a bounded linear operator. Find T^* and show that $T + T^*$ is a projection operator of rank one.

6. (10) Let A be a unital Banach algebra and $\{a_n\}_{n=1}^{\infty}$ be a sequence of invertible elements converging to some a in A. Show that if $\{||a_n^{-1}||\}_{n=1}^{\infty}$ is a bounded sequence, then a is also invertible.